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Anomalous diffusion in random media of any dimensionality

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Résumé. — Nous montrons que la diffusion est anormale en toute dimension dans les milieux aléatoires où les forces présentent des corrélations à longue portée. Ce résultat est obtenu à la fois par des arguments physiques et par une analyse de groupe de renormalisation. Les comportements obtenus sont en général surdiffusifs sauf lorsque la force aléatoire dérive d'un potentiel. Dans cette situation on obtient un comportement sous-diffusif. Dans le cas critique supérieur ($D = 2$ pour des corrélations à courte portée), celui-ci est caractérisé par un exposant dépendant continûment du désordre. La raison en est l'annulation de la fonction β qui est démontrée à tous les ordres de la théorie des perturbations. Dans le cas général, des arguments simples suggèrent qu'une force potentielle conduit à des diffusions logarithmiques (c'est-à-dire à du bruit en $1/f$).

Abstract. — We show, through physical arguments and a renormalization group analysis, that in the presence of long-range correlated random forces, diffusions is anomalous in any dimension. We obtain in general surdiffusive behaviours, except when the random force is the gradient of a potential. In this last situation, with either short or long-range correlations, a subdiffusive behaviour with a disorder dependent exponent is found in the upper critical case ($D = 2$ for short-range correlations). This is because the β-function vanishes, which is explicitly proven at all orders of the perturbation theory. Apart from this case, a potential force is expected to lead to logarithmic diffusion ($1/f$ noise), as suggested by simple arguments.

1. Introduction.

Random walks in random media (RWRM) are idealized models of a considerable number of physical situations. Examples are provided by the hopping conductivity of disordered materials [1], the diffusion of a test particle in a porous medium [2] or in turbulent flows. More abstractly, perhaps also more loosely, they could capture the essential features of the time evolution of a disordered system in its complicated configuration space, for example a spin-glass.

In its one-dimensional version, the model is by now well understood [3-7] and exhibits quite a rich variety of behaviours. Imagine a particle on a line, submitted to a Langevin thermal noise and to a quenched random force independently chosen at each point. If this force is of *zero mean* it has been shown (1) that the particle diffuses extremely slowly : its mean squared position increases logarithmically in time instead of linearly. This can be understood through an Arrhenius argument : in one dimension, a force is always the gradient of a potential, which in our case increases typically like the square root of the distance (as it is the sum of independent random

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(1) This result is due to Sinaï [5] in the case of a walk on a lattice. Explicit results for the continuous case mentioned here can be found in [7].

variables). Thus the typical time needed to travel a distance x is given by $t \sim \exp(\sqrt{x})$, which leads to $\sqrt{\bar{x}^2} \sim \ln^2 t$. If the mean of the force is non-zero, one obtains a large variety of behaviours, like for example $\bar{x} \sim t^{\mu}$, where μ is an exponent directly related to the value of the mean force.

It was soon realized that such a logarithmic diffusion was tantamount to a $1/f$ spectrum of the fluctuations and thus that *if* logarithmic diffusion could occur *in higher dimensions,* this mechanism would be a general and rather convincing source for the ubiquitous $1/f$ noise [8]. Unfortunately, it has been shown [9, 10] that if the random force is chosen independently from site to site, the diffusion is normal (i.e. $\bar{x}^2 \sim t$) for any dimension larger than two. The same remark holds when the force is divergence free (hydrodynamical flow, magnetic field) or is the gradient of a potential (Hamiltonian motion) [2, 11, 12], provided correlations remain short-ranged (see however the last paragraph of this paper). Renormalization group (RG) analysis can be performed (as first shown in [10, 13]) to obtain the dynamical behaviours in dimension $D = 2 - \varepsilon$.

This paper has two purposes. The first one is to emphasize that RWRM can display anomalous diffusion laws in *any dimension* provided the disorder has long-range correlations, and to present the results of a RG analysis of this case. We shall point out, in the next section, the physical relevance of such correlations, which are essential to the arguments of reference [8] but have not been analysed so far through RG techniques (except, as a byproduct of the study of the « TSAW », in the special case of an unconstrained force [13]). The second purpose concerns the case where the random force is the gradient of a potential, with either short or long-range correlations. It has been remarked [11, 12] that in this case the RG fails (up to two loops) to predict the dynamical behaviour. We will make a step towards the complete solution of this problem by an all-orders analysis of perturbation theory. We are motivated in particular by the recent mathematical results of Durrett [14] who finds logarithmic diffusions in a long-range correlated potential case. We will show that, contrarily to his claim, this result can very well be understood through a RG argument.

2. The relevance of disorder.

We consider a particle submitted to a thermal noise $\boldsymbol{\eta}(t)$ and to a *quenched* random force $\mathbf{V}(x)$. Its motion is described by a viscous limit of the Langevin equation (such that the velocity is instantaneously adjusted to the local force) :

$$\dot{\mathbf{x}} = \mathbf{V}(x) + \boldsymbol{\eta}(t) \tag{1}$$

with

$$\overline{\eta_i(t)\, \eta_j(t')} = 2\, D^0\, \delta(t - t')\, \delta_{ij}$$

$$\langle \mathbf{V}(x) \rangle = \mathbf{V}^0, \quad \langle V_i(x)\, V_j(y) \rangle_{\text{conn}} = G_{ij}(x - y)\,. \tag{2}$$

The constraints on $\mathbf{V}(x)$ determine the (ij) structure of G_{ij} ; this is more easily expressed in Fourier transform by defining :

$$G_{ij}(k) = G_{\mathrm{T}}(k^2)(\delta_{ij} - k_i\, k_j/k^2) + G_{\mathrm{L}}(k^2)\, k_i\, k_j/k^2 \tag{3}$$

we will mainly consider three models :

(model I) unconstrained force : $G_{\mathrm{L}} = G_{\mathrm{T}}$
(model II) divergenceless flow : $G_{\mathrm{L}} = 0$
(model III) potential case $\mathbf{V} = -\operatorname{grad}(H)$: $G_{\mathrm{T}} = 0$.

We define an exponent a characterizing the spatial decay of the correlations :

$$\begin{aligned} G_{\mathrm{L}}(b(x-y)) &\underset{b \to \infty}{\sim} b^{-a}\, G_{\mathrm{L}}(x-y)\,, \\ G_{\mathrm{T}}(b(x-y)) &\underset{b \to \infty}{\sim} b^{-a}\, G_{\mathrm{T}}(x-y)\,. \end{aligned} \tag{4}$$

Power law correlations can be of great physical relevance, for example to the diffusion of a test particle in a steady flow accross a porous medium (model II). Indeed, the flow can be highly inhomogeneous and nevertheless display preferential paths along which correlations remain strong. Another interesting application lies in spin-glass dynamics (to which model III could be relevant). In this context, one has the intuitive idea that energy barriers between two configurations should be an increasing function of their distance. In the Sherrington-Kirkpatrick [15] infinite-range model for example, a simpler quantity, such that the correlation $\langle (H(\{S_i\}) - H(\{S_i'\}))^2 \rangle$ is easily shown to be an increasing linear function of the « distance » between the two configurations $\{S_i\}$ and $\{S_i'\}$, in a wide range of distances. This corresponds to a correlation between forces with $a < 2$. We wish to emphasize that, physically, (4) needs not hold on very large scales $(x - y)$ when diffusion is ultra-slow, but only in the restricted region of space that the diffusing particle can probe within a given time. In the case of a logarithmic diffusion for example this defines a frequency cut-off exponentially small in the length and in temperature : $f_{\mathrm{c}} \sim \exp(-L/T)$ down to which $1/f$ noise will be observed. Let us mention finally that explicit construction of fractal energy landscapes can be given [16], for which $\langle H(x)\, H(y) \rangle = |x - y|^{2-a}$ at all scales, with $a - 2 = 2(D - D_{\mathrm{F}}) < 0$, D_{F} being the fractal dimension of the landscape.

We now show that the region in the plane (a, D) where disorder is relevant (in the sense that it leads to an anormal diffusion law and not only changes the diffusion coefficient) can be inferred by a simple argument. Suppose first that there are no correlations. In the pure case ($\mathbf{V}=0$) one has $\overline{x^2(t)} \sim t^{2\nu_0}$ with $\nu_0 = 1/2$. The number of sites which are visited by the walker in a time t is thus $\inf(t, t^{\nu_0 D})$, equal to t if $D\nu_0 > 1$, that is for $D > 2$. Reasoning perturbatively, it means that when a small amount of disorder is present in $D > 2$, the walker essentially probes of the order of t *different* values of $\mathbf{V}$, which according to the central limit theorem, results in an *apparent* mean value of the velocity given by :

$$V_{\mathrm{av}} = \frac{1}{t}\sum_{i=1}^{t} V_i \simeq \frac{\sqrt{\sigma t}}{t}. \qquad (5)$$

The extra displacement due to the disorder is thus given by :

$$\Delta x \simeq V_{\mathrm{av}}\, t \simeq \sqrt{\sigma t}$$

leading to

$$X^2 = D^0 t + \sigma t \,,$$

which implies that, for an ordinary random walk in $D > 2$, short-range correlated disorder simply changes the value of the diffusion coefficient (2) ($\delta D^0 \simeq \sigma$). Below this dimensionality, the walker is extremely sensitive to local fluctuations of the force $\mathbf{V}$, since in the pure case it visits infinitely often a given site. This can lead either to subdiffusive behaviours (if trapping regions dominate) or to surdiffusion (if locally ballistic motion is preserved, like in model II, see [11]).

One can discuss along the same lines the case where there are long range correlations of type (4). (4) means that at a distance R from a point O one roughly finds $\int^R r^{D-1}\,\mathrm{d}r/r^a$ sites carrying the *same* value of $\mathbf{V}$ as O. For large R, this integral behaves as a constant if $a > D$ and as R^{D-a} if $a < D$. In a sphere of size R^D, one can thus organize sites in sets of size R^{D-a} inside which $\mathbf{V}$ takes roughly the same value. The number of *independent values* of the disorder is therefore $R^D/R^{D-a} = R^a$. In other words, the long range correlations have reduced the dimensionality of space to an « effective » one, a, to which the above argument on residual average velocity can be applied. Thus, for $a > 2$, the disorder simply modifies the diffusion coefficient, while for $a < 2$ one expects anomalous diffusions *independently* of the dimension D : the resulting domain in the (a, D) plane where disorder is relevant is shown in figure 1. For model III, the analysis has to be slightly modified, and leads to the conclusion that disorder is only relevant if $a < 2$, independently of D, i.e. in the case where correlations of the *potential* increase with

(2) Note that this also means that for Levy flights ($\nu_0 > 1/2$), Gaussian disorder is irrelevant in $D > 1/\nu_0$.

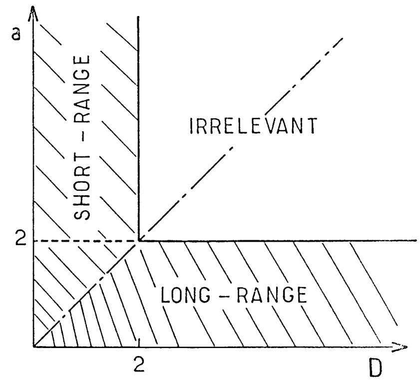

Fig. 1. — The region of the plane (a, D) where disorder is relevant is shown by hatches. In the potential case, only the (dotted) line $a = 2$ remains. Previous studies of the short range correlated case span the (dashed-dotted) line $a = D$.

the distance. The power-counting analysis developed below will confirm these conclusions rigorously : in the field-theoretical framework, our qualitative argument can be seen as a « Harris criterion » (whose modification in long-range correlated cases has been studied in [17]).

3. Renormalization group analysis.

The field-theoretical analysis starts from the Fokker-Planck equation for the probability density $P(x, t)$ associated with (1). The average of P over the random force (taken to be Gaussianly distributed according to (2)) can be expressed, using the replica trick, as the 2-point function of a field theory involving 2 N scalars fields ϕ^a, $\bar{\phi}^2$:

$$\langle P(x, \omega)\rangle = \lim_{N\to 0}\frac{1}{N}\langle \phi^a(0)\,\bar{\phi}^a(x)\rangle_s \qquad (6)$$

where a Laplace transform in time has been taken. The action S reads :

$$S = \int \mathrm{d}^D x \left\{ D^0\,\partial_i\bar{\phi}^a\,\partial_i\phi^a - 2\,V_i^0\,\phi^a\,\partial_i\bar{\phi}^a + \right.$$
$$\left. + \,\omega\phi^a\,\bar{\phi}^a\right\} + 2\,i\int \mathrm{d}^D x\,\mathrm{d}^D y\;\phi^a(x)\,\partial_i\bar{\phi}^a(x)\,\times$$
$$\times\, G_{ij}(x-y)\,\phi^b(y)\,\partial_j\bar{\phi}^b(y)\,. \qquad (7)$$

In the static limit ($\omega = 0$), this field theory has a critical point at $\mathbf{V}=0$. This is related to long-distance singularities arising when $\langle \mathbf{V}_0\rangle \to 0$ in the weak-disorder expansion of lattice hopping models with asymmetric hopping rates. Indeed, it can be shown [10] that (7) reproduces the dominant $I \,.\, R$ singularities of a large class of hopping models (see, however, last paragraph of this paper). In one dimension, which is the lower critical one for model

(7), the critical *point* at $\mathbf{V}^0 = 0$ is replaced by a whole domain of V^0 where the velocity $\overline{x(t)}/t$ vanishes at $t \to \infty$ (critical *phase*) [6].

The exponent ν characterizing the long-time behaviour of $\overline{x^2(t)}$ can in fact be related to the exponent η of model (7) in the purely static limit $\omega = 0$. Indeed, probability conservation implies that the susceptibility exponent γ is exactly equal to 1 (since $\omega P(\mathbf{q} = 0, \omega) = 1$). The scaling relation $\gamma = \nu(2 - \eta)$ thus leads to $\overline{x^2(t)} \sim t^{\frac{2}{2-\eta}}$ where η is the usually defined exponent characterizing the algebraic decay of the 2-point function at criticality in the static theory ($\omega = 0$, $\mathbf{V}^0 = 0$).

The superficial $I.R$ degree of convergence of a diagram with $E/2\ \phi$ (and $E/2\ \bar{\phi}$) — external legs at the order n of perturbation theory is easily shown to be :

$$\omega_{I.R} = D + E(1 - D)/2 + n\,(D - 4 + \omega_{\mathrm{V}}) \tag{8}$$

ω_{V} is the momentum dependence of the vertex, arising from the derivative couplings and from the momentum dependence of the non-local interaction. Owing to (4), we see that $G_{\mathrm{T,L}}(k^2)$ behaves (when $k^2 \to 0$) as a constant, $\sigma_{T,L}$, if $a > D$ and as $\sigma_{\mathrm{T,L}}(k^2)^{(D-a)/2}$ if $a < D$. This leads to : $\omega_{\mathrm{V}} = 2$ if $a > D$ and $\omega_{\mathrm{V}} = 2 + a - D$ if $a < D$. We deduce from (8) that the upper critical « dimension » is $D = 2$ if $a > D$ and $a = 2$ if $a < D$: this confirms the analysis of the preceeding section, summarized in figure 1. Renormalizability is likely [10] in the upper critical case (for $\omega = 0$), despite the non-local character of the model and the fact that an infinite number of dimensionnally admissible counterterms *a priori* exists. This is due to global symmetries ($U(N)$, $\bar{\phi}^a \to \bar{\phi}^a + \mathrm{Cst.}$) and to the fact that non-locality arises only from the « non-canonical » kinetic term of the force V_i, seen as an auxiliary field. A non zero ω could nevertheless spoil these renormalizability properties.

The one-loop RG analysis can be performed along the lines of reference [18] by working in the framework of the renormalized theory. Defining dimensionless coupling constants g_{T} and g_{L} through : $\sigma_{\mathrm{L}} = g_{\mathrm{L}}\,\mu^{2-a}$ and $\sigma_{\mathrm{T}} = g_{\mathrm{T}}\,\mu^{2-a}$ at the tree level, we obtain at one loop order :

$$\beta_{\mathrm{T}} = \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,g_{\mathrm{T}} = -\,\varepsilon g_{\mathrm{T}} + 4\,C_D\left(1 - \frac{1}{D}\right) \times g_{\mathrm{T}}(2\,g_{\mathrm{T}} - g_{\mathrm{L}})$$

$$\beta_{\mathrm{L}} = \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,g_{\mathrm{L}} = -\,\varepsilon g_{\mathrm{L}} + 4\,C_D\left(1 - \frac{1}{D}\right) g_{\mathrm{L}}\,g_{\mathrm{T}}$$

$$\eta = \mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\frac{D}{D^0} = -\,4\,C_D\left[\left(\frac{1}{D} - 1\right) \times g_{\mathrm{T}} + \frac{1}{D}\,g_{\mathrm{L}}\right] \tag{9}$$

where ε denotes

$$2 - D, \quad C_D = 1/2 \qquad \text{if} \quad a > D$$

$$\text{and } \varepsilon = 2 - a, \quad C_D = \frac{2^{1-D}\,\pi^{-D/2}}{\Gamma\left(\frac{D}{2}\right)} \qquad \text{if} \quad a < D\,.$$

Models I, II and III thus appear as remarkable cases whose structure is preserved by renormalization. If one starts with a bare G_{ij} which is neither of type II nor of type III, the one-loop RG flow goes to the fixed point $g_{\mathrm{T}}^* = g_{\mathrm{L}}^* = \varepsilon/4\,C_D(1 - 1/D)$: thus a « generic » RWRM (without constraints II or III) is in the universality class of model I. Starting from model II leads to the fixed point $g_{\mathrm{T}}^* = \varepsilon/8\,C_D(1 - 1/D)$, $g_{\mathrm{L}}^* = 0$ (for which $g_{\mathrm{L}} = 0$ is an attractive direction, but which is repulsive otherwise). For these two cases, the long time behaviour of $\overline{x^2(t)}$ resulting of this one loop-analysis is given in table I for both short-range and long-range correlations. (In the upper critical cases, a direct integration of the one loop RG equations has been performed). In the short range case, we recover the results of reference [2, 9-12] (a two-loop computation of η is then necessary for model I.) The behaviour $V = (V^0)^{\varphi}$ of the renormalized velocity as a function of V^0 is also given. The exponent is related to the RG function γ_{V} associated with insertions of the operator $\phi^a\,\partial_i\bar{\phi}^a$ through $\varphi = 1/(1 - \gamma_{\mathrm{V}})$. γ_{V} is related to vertex-type corrections of the 4-point function, and reads at one-loop order :

$$\gamma_{\mathrm{V}} = 4\,C_D\left(1 - \frac{1}{D}\right) g_{\mathrm{T}}\,. \tag{10}$$

Table I.

MODEL		I	II
Short range correlated case : ($\varepsilon = 2 - D$)			
$D = 2$	$\overline{x^2(t)}$	$t\left(1 + \frac{4}{\ln t}\right)$	$t\sqrt{\ln t}$
$D < 2$	$\overline{x^2(t)}$	$t^{1-\varepsilon^2}$	$t^{1+\frac{\varepsilon}{4}}$
	$V = (V^0)^{\varphi}$	$\varphi = 1 + \varepsilon$	$\varphi = 1 + \frac{\varepsilon}{2}$
Long range correlated case : ($\varepsilon = 2 - a$)			
$a = 2$	$\overline{x^2(t)}$	$t(\ln t)^{\frac{D-2}{D-1}}$	$t\sqrt{\ln t}$
$a < 2$	$\overline{x^2(t)}$	$t^{1+\frac{D-2}{D-1}\frac{\varepsilon}{2}}$	$t^{1+\frac{\varepsilon}{4}}$
	$V = (V^0)^{\varphi}$	$\varphi = 1 + \varepsilon$	$\varphi = 1 + \frac{\varepsilon}{2}$

MODEL III

$a > 2 \quad \overline{x^2(t)} \sim t$

$a = 2 \quad \overline{x^2(t)} \sim t^{1 - c_D \sigma}$

$a < 2 \quad \overline{x^2(t)} \sim (\ln t)^{\frac{4}{2-a}}$

The fact that model II always leads to *surdiffusion* can be understood (as noticed in [11]) from the existence of closed lines of force along which ballistic motion is preserved. On the other hand, introducing long-range correlations in model I turns the subdiffusive behaviour into a surdiffusive one.

Model III is extremely peculiar from the RG point of view, since no contribution to the β function (and thus no fixed point) is found at one-loop order. Explicit calculations in the short range case [11] reveal that this property is maintained at two loops. The rest of this paper is devoted to a discussion of this model.

4. The potential case (Model III).

We will first *show* that for this model one has $\beta_L = \varepsilon g_L$ without corrections at any order of perturbation theory. Arguments in favour of this point in the short range case have been given in [11]. Setting $\mathbf{V} = -\operatorname{grad}(H)$, we choose not to perform explicitly the Gaussian average leading to (7) and to keep instead H as an auxiliary field. In the short range case, where $G_L(k^2)$ is a constant, the action S is thus replaced by a *local* field theory whose Lagrangian reads : $(\sigma \to i\sigma)$

$$\mathfrak{L} = -\frac{1}{8\sigma}(\partial H)^2 + D^0\,\partial_i\bar{\phi}^a\,\partial_i\phi^a + \partial_i H\,\phi^a\,\partial_i\bar{\phi}^a . \tag{11}$$

We recognize a non-linear σ-model [19] for the $2N+1$ fields $(\phi^a, \bar{\phi}^a, H)$. Computing the Ricci and curvature tensor leads to $R_{ab} \propto -N g_{ab}$ and $R_{abcd} \propto (g_{ac}\,g_{bd} - g_{ad}\,g_{bc})$, with $\det(g_{ab}) = 1$. Surprisingly enough, we discover that the model has a hidden non-linear $O(N+1, N+1)$ non-compact symmetry. (We have found in explicit form the transformation on the fields which allows to express (11) in the standard form of a $O(N+1, N+1)$ field theory.) When the $N \to 0$ limit of the replica trick is performed, one is thus left with a free theory for which $\beta_L = -\varepsilon g_L$ at all orders. In the long range correlated case, the σ-model becomes non local in space, but the same conclusion holds (since it comes only from an « internal symmetry » property).

In the upper critical case (short or long-ranged, $\varepsilon = 0$), the function β_L is zero ; the running coupling constant $g(\lambda)$ thus remains equal to the bare one at any scale λ. This leads to a continuous dependence of the exponents on the variance of the disorder (in this case, the results of Tab. I are valid to all orders).

When $\varepsilon > 0$, $g(\lambda)$ is driven to infinity in the infrared limit $\lambda \to 0$: $g(\lambda) \sim g_0\,\lambda^{-\varepsilon}$. This means that it is the *strong disorder* regime which controls the physics, regardless of the value of g_0. In this limit, particles are trapped for extremely long times in the bottom of the potential wells, where we expect the probability distribution $P(x,t)$ to be well approximated by its equilibrium value which is the Boltzmann weight $\exp(-H(x))$. This distribution behaves *typically* as $\exp(-g^{1/2})$ at large g, and inserting $g(\lambda) \sim \lambda^{-\varepsilon}$ this leads to a long distance behaviour $\exp(-x^{\varepsilon/2})$. This suggests (requiring that $tP(x)$ should be of order 1) a logarithmic diffusion (see Tab. I).

This result is the one that can be guessed from an Arrhenius argument similar to the one given in the introduction. In $D = 1$, this argument is certainly correct, since there is only one path going from one point to another. In higher dimensions, diffusion between a point O and any point A at a distance R will be dominated by the smallest energy barrier encountered by paths going from O to A and restricted to a volume R^D. As we are interested in most efficient paths, they can of course be taken as self-avoiding. Now, on any path of internal length N, the typical energy barrier one encounters scales like $\sqrt{N}$ (we deal here with short-range correlations such that $a = D$) ; nevertheless the minimal energy barrier between p independent paths is obviously $\sqrt{N/p}$. In a sphere of radius R, one has roughly R^D/N independent paths of size N, and as the paths are self-avoiding one also has $R = N^{\nu_{SAW}}$. Therefore, we predict that the minimal energy barrier scales in this system as $R^{[3/2\,\nu_{SAW} - D]}$. Diffusion is thus logarithmic if $\nu_{SAW} < 3/2\,D$, i.e. if $D = a < 2$, as $\nu_{SAW}(D=2) = 3/4$. Inserting the Flory value (exact in $D = 1, 2, 4$), one obtains $t = t_0\exp(R^{(2-D)/2})$, and thus $R = (\ln t/t_0)^{2/(2-D)}$ which is, quite surprisingly, exactly the result obtained through the RG argument presented above. In $D > 2$, the great number of paths always allow the walker to go around « hills ». In the general case of long-range correlations, the condition $a < 2$ means that the potential grows at large distances like $H(bx) = b^{(2-a)/2}H(x)$ and the Arrhenius argument allows to recover the behaviour of table I. Lastly, we would like to point out that our model is a continuous limit of the discrete one for which Durrett [14] has *proven* that this logarithmic diffusion takes place in any dimension. Indeed, if one identifies

$$-\nabla H(x) = \lim_{x \to x'} \sum_{x'} (W_{xx'} - W_{x'x})(x' - x)$$

where $W_{xx'}$ is the hopping rate from site x to site x', one gets exactly with Durret's definition : $\mathbf{V} = -\operatorname{grad}(H)$, with $H(bx) = b^{\alpha}H(x)$, his exponent α being $(2-a)/2$.

Let us finally mention that not all RWRM fall in the universality classes of table I. For example, random walks among traps with a broad distribution of release time τ of the type $\rho(\tau) \sim \tau^{-\alpha}$ (with $\alpha < 2$) reveals subdiffusive behaviour in all dimensions with an exponent $\nu = \dfrac{\alpha - 1}{2}$ for $D > 2$ [20]. We believe that the example recently constructed in

[21] falls in this universality class rather than in those studied in this paper : the potential constructed in [21] has short-range correlations with a local distribution of potential depths $\rho(H) \sim e^{-aH}$. Upon rescaling of space, this induces traps with a release time given by Arrhenius law $\tau \sim \exp(\beta H)$. Its distribution is thus given by $\tau^{-\left(1+\frac{a}{\beta}\right)}$, leading us to conjecture that the diffusion law for this model is $\bar{x}^2 \sim t^{a/\beta}$, for $D > 2$.

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